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THE MATHEMATICS TEACHER

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NON-EUCLIDEAN GEOMETRY[1]

By DR. W. H. BUSSEY
University of Minnesota

About 2200 years ago there was published in Greek one of the most remarkable books of all times, Euclid's "Elements of Geometry". It contains a systematic exposition of the leading propositions of elementary geometry and the elementary theory of numbers. It was at once adopted by the Greeks as the standard text book on pure mathematics. The parts that relate to elementary geometry were the standard text book for centuries and are still in use in England to-day. The English school boy does not say "Geometry", he says "Euclid". On the Continent of Europe "Euclid" was superseded by Legendre's "Elements of Geometry", the first edition of which was published in 1794. A translation into English by a man named Davies was widely used in this country. (It was used at Columbia University as late as 1905). But that has been superseded by more modern American texts of which there is now a large number.

The best known English edition of Euclid is probably that of Todhunter which contains Books I to VI and parts of Books XI and XII. The most complete and the most valuable edition is "The Thirteen Books of Euclid's Elements" by T. L. Heath in three volumes with many pages of historical, critical and explanatory notes.

An edition by Robert Simson of the University of Glasgow was published in Philadelphia in 1811. The title page says that "the errors by which Theon and others have long ago vitiated these books are corrected and some of Euclid's demonstrations are restored". On the inside of the cover of a copy of this book I found written in pencil: "Q. E. D. Quid Euklid Dixit," an anonymous tribute to the authority of Euclid.

[1] Paper read before the Twin City Mathematics Club, May 13, 1922.

Euclid's work on geometry is largely a compilation from the works of previous authors. Just how much of it was original with Euclid is unknown. Many of his theorems are known to have been due to Thales, Pythagoras, Hippocrates and others. But the material he took from other authors was rearranged by him and in some cases new proofs were substituted.

Euclid's book begins with definitions, postulates, and common notions, some of which I shall quote.

Definitions:

1. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Common Notions:

1. Things which are equal to the same thing are equal to each other.
2. The whole is greater than the part.

Postulates:

1. A straight line can be drawn from any point to any point.
2. A finite straight line may be produced continuously in a straight line.
3. All right angles are equal.
4. A circle can be described with any centre and any radius.

These common notions and postulates are often referred to as "axioms".

There is another axiom of Euclid's usually known as Postulate 5, which differs from the ones I have quoted and from all the others. It is prolix. It has different places in the different manuscripts. It is not used in the first 28 propositions and then it is used only to prove the converse of a proposition previously proved. It sounds more like a proposition than an axiom. Euclid proved propositions far more self-evident. As someone has said, he proved what every dog knows that two sides of a triangle are, taken together, greater than the third side. Yet

when he had proved Proposition 28, that lines making equal angles with a transversal are parallel, in order to prove the converse he stated and used this postulate:

Postulate 5: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side on which are the angles less than two right angles".

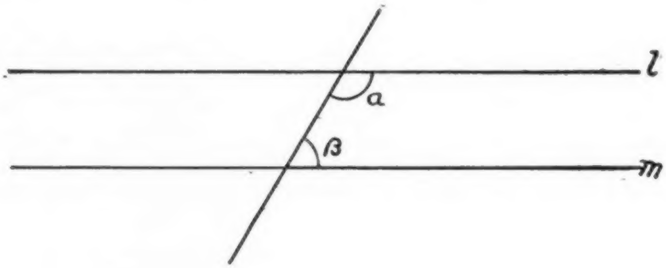
Perhaps you are not familiar with it in this awkward form. It is equivalent to this: "Two straight lines which intersect one another can not both be parallel to the same straight line", or "Through a given point only one parallel can be drawn to a given line". This is commonly known as Playfair's form of the parallel axiom although it was known to Proclus, who wrote a commentary on Euclid's Elements in the 4th century A. D.

Euclid proved in Book I, Proposition 28 that through a point there is one parallel to a given line. Playfair's axiom states that there is only one. With this assumption Euclid's Postulate 5 is easily proved, and vice versa.

I have said that Postulate 5 sounds more like a proposition than an axiom. To make this clear, I shall quote it again: "If a straight line falling on two straight lines makes the interior angles on the same side together equal to two right angles, the two straight lines will be parallel to one another". That sounds very much like what I quoted before as Postulate 5, but as a matter of fact it is no such thing. It is Proposition 28. Euclid proved it and all the previous propositions without using Postulate 5, which I shall now read again: "If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side on which are the angles less than two right angles".

He used it in the proof of Proposition 29 which is as follows: "If a straight line fall on two parallel straight lines, it makes the interior angles on the same side of the line together equal to two right angles". You must keep in mind Proposition 28,

Postulate 5 and Proposition 29. To this end I shall state all three very briefly in terms of a single diagram.



Prop. 28: If $\alpha + \beta = 180°$, lines l and m are parallel.
Postulate 5: If $\alpha + \beta < 180°$, lines l and m meet.
Prop. 29: If lines l and m are parallel, $\alpha + \beta = 180°$.
Remember that Euclid proved proposition 28 and then had to assume Postulate 5 in order to prove Proposition 29.

From the very beginning Postulate 5 was attacked as not self-evident and as needing demonstration, and Euclid's contemporaries and successors tried to prove it by reasoning from the other exioms. Proclus in the 4th century A. D. said: "This ought to be struck out of the postulates altogether for it is a theorem involving many difficulties which Ptolemy, in a certain book, set himself to solve and it requires for the demonstration of it a number of definitions as well as theorems. And the converse of it is actually proved by Euclid as a theorem":

The converse of it, in terms of the diagram given above is:
"If lines l and m meet, $\alpha + \beta < 180°$."

It is equivalent to Proposition 28 which is:
"If $\alpha + \beta = 180°$, lines l and m are parallel."

We have a record of noteworthy attempts to prove Postulate 5 dating from Ptolemy's attempt in the 2nd century A. D. to the present day; with others still to come although they will not be noteworthy because it has been demonstrated by mathematicians that the thing can not be done. All the attempts were failures in the sense that either they begged the question, or else they assumed something else equivalent to Postulate 5 and just as much in need of proof.

Geminus (about 50 B. C.) assumed that there exist straight lines everywhere equidistant from one another.

Ptolemy (2nd century A. D.) really begged the question.

Proclus (5th century A. D.) assumed that if a straight line intersects one of two straight lines it intersects the other also.

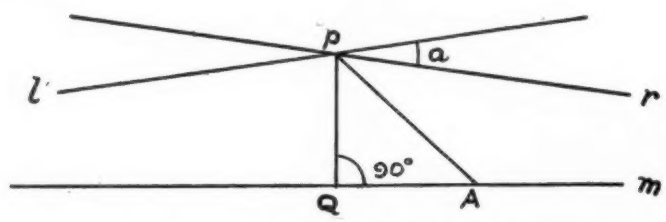
Wallis (17th century A. D.) assumed that, given any figure, there exists a similar figure similar to it of any size we please.

Legendre (1752-1833) was perhaps the greatest mathematician who, up to his time, attempted to prove Postulate 5. His different attempts appeared in the successive editions of his "Elements of Geometry" from the first in 1794 to the twelfth in 1823. His exposition brought out clearly the essential connection between the axiom of parallels and the sum of the angles of a triangle. In the first edition of his geometry the proposition that the sum of the angles of a triangle is equal to two right angles was proved analytically on the basis of the assumption that the choice of a unit of length does not affect the correctness of the proposition to be proved, which is of course equivalent to Wallis' assumption of the existence of similar figures. In the second edition he proved Postulate 5 by means of the assumption that, given three points not in a straight line, there exists a circle passing through all three. In the third edition he proved that the sum of the angles of a triangle is not greater than two right angles. But he could not prove the angle sum to be not less than two right angles. If he had succeeded in that, he would have proved the equivalent of Postluate 5, namely that the angle sum in a triangle is exactly equal to two right angles. Legendre proved this, however: "If the sum of the angles of *one* triangle is equal to two right angles, the sum of the angles of *any other* triangle is also equal to two right angles.

In 1826, a few years before Legendre's death, a Russian named Lobatschewsky published at Kasan, in Russia, a little book called "The Theory of Parallels." It was followed in 1836, 1837 by other papers on the same subject. His work was unnoticed for years. In 1840 he published a resume of it in German and in 1855, a year before he died, he published a complete exposition of it in French and Russian. As early as

1815 he was working on parallels and he made several attempts to prove Postulate 5 and some investigations resembling those of Legendre. I do not know what was in his mind when he published the booklet in 1826, but I like to think that he was trying to prove Postulate 5 by the reductio ad absurdum method. If the postulate could be proved by logical reasoning, from Euclid's other axioms, a denial of the postulate ought to lead to a contradiction and thus result in a proof by the so called indirect method. At any rate, Lobatschewsky denied that there was only one line through a point and parallel to a given line; he assumed that there was more than one. (Remember that Euclid proved the existence of one). Lobatschewsky did it in this way:

Let m denot a line, A a point on that line, and P a point not on that line.



Let A move indefinitely to the right and denote by r the line which is the limiting position approached by the line PA. In like manner let l be the limiting position approached by PA as A recedes indefinitely to the left. And assume that r and l are different lines. (On the Euclidean assumption, r and l are one and the same line.) The figure consisting of P, m, r, l is symmetrical with respect to the line PQ drawn from P perpendicular to m. It is evident that any line through P which lies within the angle (a) between r and l will not meet m. The number of such lines is infinite. So with the Euclidean definition of the word "parallel," Lobatschewsky's assumption really is that the number of parallels through P is not *one* but *infinite*. But Lobatschewsky used the word "parallel" with a somewhat different meaning. With reference to P and m, he called (r) a parallel on the right and (l) a parallel on the left. He classified the lines through P, in their relation to m, as "intersecting

lines," "non-intersecting lines," and "parallels" defined as the boundary lines between the "intersecting" and the "non-intersecting" lines.

Now if he had this substitute for the Euclidean axiom in mind as the beginning of a reductio ad absurdum proof of Postulate 5, he naturally would have tried to deduce logical consequences of his hypothesis and the Euclidean assumptions other than Postulate 5 with the expectation that sooner or later he would arrive at a *contradiction* which would be the end of the reductio ad absurdum proof. And the result would have been an indirect proof of Postulate 5. Whatever his expectation was, he never arrived at such a contradiction. He proved theorem after theorem; he pointed out that many of Euclid's theorems were just as valid with his assumption about parallels as with Euclid's assumption. He finally came to the conclusion that Euclid's Postulate 5 could not be deduced as a logical consequence of Euclid's other axioms.

If one is trying to make a reductio ad absurdum proof and fails to find the contradiction which he seeks, he is not justified in saying that no such contradiction can be found. If he has tried for 5 hours without result, he may find the contradiction in the next five minutes, and if he gives it up as a bad job, he can never be sure that he would not have succeeded if he had persevered. So the fact that Lobatschewsky did not find a contradiction did not prove that there was none to be found. It remained for later mathematicians to prove conclusively that the set of theorems developed by Lobatschewsky together with all others logically deducible from his assumptions constitute as logically consistent a geometry as the Euclidean Geometry known to you all. Cayley, an Englishman, and Klein, a German are jointly responsible for the demonstration of this fact. The details of their work are long and they involve much higher mathematics. I shall not speak of them further.

Now you may be interested in knowing some of the characteristics of this geometry of Lobatschewsky. Propositions whose proofs depend solely on the method of superposition are the same as in Euclidean Geometry. Here are some of the more important theorems that are Non-Euclidean.

1. The sum of the angles of a triangle is less than 180°. The amount by which the angle sum is less than 180° is called the "deficiency of the triangle".

2. The areas of two triangles are to each other as their deficiencies. As the area of a triangle becomes smaller, the deficiency approaches zero as its limit or, in other words, the angle sum approaches 180° as its limit and the state of affairs approaches the Euclidean. There is a finite maximum to the area of a triangle.

3. In a tri-rectangular quadrilateral, the fourth angle is acute.

4. Parallel lines continually approach one another.

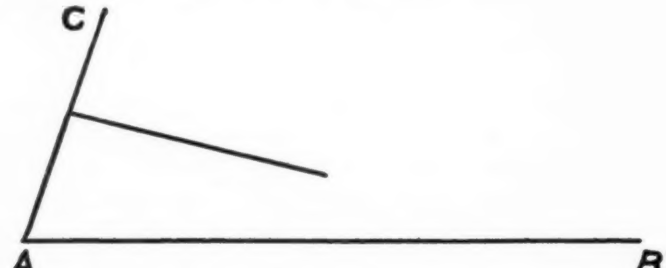
5. The angle between the line *r* and the line *PQ* in the figure given above is called the angle of parallelism for the distance *PQ*. As *PQ* increases from 0 to infinity, the angle of parallelism decreases from 90° to 0.

6. The perpendiculars erected at the middle points of the sides of a triangle are all parallel if two of them are parallel.

7. Straight lines which do not intersect and are not parallel have one and only one common perpendicular.

8. A straight line may be drawn perpendicular to a plane and parallel to any straight line not in the plane.

9. The locus of points equidistant from a straight line is not a straight line. It is called an equidistant curve.



10. Given a straight line *AB;* as its extremity *A* draw any arbitrary angle *BAC* and produce *AC* so that its perpendicular bisector shall be parallel to *AB*. The locus of *C* is called a "boundary curve" or "oricycle". *AB* is called an axis. The curve has an infinite number of such axes.

11. A boundary curve has a constant curvature. It is the limiting curve between a circle and an equidistant curve. That is, it may be thought of as a circle of infinite radius or an equidistant curve whose base line is at an infinite distance.

I have stated theorems enough to give you some idea of what this Non-Euclidean Geometry is like. I have said that its origin is due to Lobatschewsky. As a matter of fact it was developed independently by John Bolyai, a Hungarian, and it has been surmised that the same thing was done by the great German mathematician, Gauss, but he never published his results, some say because he did not want to detract from the credit due the other two, especially his friend, Bolyai.

Euclid assumed that there was only one straight line through a point and not meeting a given line. Lobatschewsky assumed that the number was greater than one and therefore infinite. You may ask why did not some one assume that there was no such line and thus develop a different Non-Euclidean Geometry. The answer is, someone did. His name is Rieman. He denied the Euclidean Postulate 5 by saying that every two straight lines meet in one and only one point. What then of Euclid's Proposition 28 which proves the existence of one straight line through a given point and parallel to a given straight line. The proof of that proposition depended on Euclid's assumption that a straight line is of infinite length. Rieman denied that too. He assumed a straight line to be "unbounded" but not "infinite". That there is a real distinction between the two, I think you will see if you think of the surface of a sphere. It is unbounded but not infinite. So the proof of Euclid's Proposition 28 fails in Rieman's Non-Euclidean Geometry. Here are some of the more important propositions of it. (Of course many theorems are the same as in Euclidean and in Lobatschewsky's Non-Euclidean Geometry.)

> 1. The sum of the angles of a triangle is greater than 180°. The amount by which the angle sum is greater than 180° is called the "excess" of the triangle.

2. The areas of two triangles are to each other as their "excesses". As the area of a triangle becomes smaller, the excess

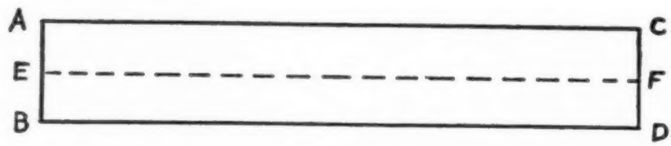
approaches zero as its limit or, in other words, the angle sum approaches 180° as its limit and the state of affairs approaches the Euclidean.

3. In a tri-rectangular quadrilateral, the fourth angle is obtuse.

4. A straight line is of finite length. It returns unto itself.

5. All perpendiculars to a given straight line meet in a point called its pole which is a half line-length away from the given line. The line is called the polar of the point.

6. A straight line does not divide the plane into two pieces. That is a line can be drawn in a plane from one side of a straight line to another without crossing the line. The plane is what is called a unilateral surface. To get an idea of what a unilateral surface is, take a strip of paper *ABCD*



and join its two ends so that *C* coincides with *B* and *D* coincides with *A*. Then cut along the line *EF* and see if the cut separates the paper into two parts.

7. The locus of points at a given distance from a straight line is a circle having its center at the pole of the line.

8. There is a trirectangular triangle.

9. A straight line is the limit approached by a circle where the length of the radius approaches one half of a line-length.

10. The plane is unbounded but not infinite.

11. All planes perpendicular to a straight line meet in a straight line called the conjugate of the given line. The relation between the two lines is reciprocal and all the points of one are a half line-length away from the other line.

12. There are straight lines not in the same plane which have an infinite number of common perpendiculars and are everywhere equidistant.

13. There are no similar figures in this Non-Euclidean Geometry.

Now you know a little something about two Non-Euclidean Geometries. I am going to suggest a third. It resembles the one I have just talked about and it has these characteristics.

1. The straight line is of finite length and returns unto itself. It is the shortest line joining two points.

2. Every two straight lines meet in two points.

3. The sum of the angles of a triangle is greater than 180°. The amount by which the angle sum is greater than 180° is called the "excess" of the triangle.

4. The areas of two triangles are to each other as their "excesses". As the area of a triangle becomes smaller, the excess approaches zero as its limit.

5. In a tri-rectangular quadrilateral the fourth angles is obtuse.

6. All perpendiculars to a given straight line meet in two points each a fourth of a line-length away from the given line. They are called the poles of the line. The line is called the polar of each of the points.

7. The locus of points at a given distance from a straight line is a circle having its centre at one of the poles of the line.

8. There is a tri-rectangular triangle.

9. A straight line is the limit approached by a circle as the length of the radius approaches one fourth of a line-length.

10. There are no similar figures in this Non-Euclidean Geometry.

You have noticed that this Non-Euclidean Geometry resembles the Rieman Non-Euclidean Geometry, but that there are important differences, namely these. In this geometry last mentioned, two straight lines meet in two points, not one; the distance from a straight line to its polar is one fourth and not one-half of a line-length; a straight line *does* divide the plane into two parts; the plane is *not* unilateral. In other respects it resembles the Rieman Non-Euclidean Geometry. But in a very real sense it is Euclidean Geometry, and it is well known to all of you. If you will substitute "sphere" for "plane" and "great circle" for "straight line" and "spherical excess" for "excess," it will become what is known as spherical geometry, that is, the geometry of great circles on the surface of a sphere. If you

have difficulty in thinking of great circles on a sphere as analogus to straight lines in a plane, remember that the great circle is the shortest line on a sphere connecting two points of the sphere, and the great circle is the line traced out by a point on the sphere which starts in a given direction and turns neither to the right nor to the left. And if geometry had been first developed for purposes of navigation on the great oceans, spherical geometry might have been developed before Euclidean plane geometry, just as spherical trigonometry was developed before plane trigonometry.

Perhaps you are now inclined to ask: "What is the significance of these Non-Euclidean Geometries and what is their relation to the space in which we live?" The easy answer to give is that Non-Euclidean Geometry is a branch of pure mathematics developed by mathematicians to satisfy their intellectual curiosity and that it has no relation to the space in which we live. But that is not the only answer that has been given to the question.

Geometry used to be thought of by almost everybody as an exact science having absolute certainty, as a science developed by logical reasoning from a set of unalterable axioms which were self-evident and a part of our intuition of space. The idealists of the 17th and 18th centuries, who held that certain knowledge, independent of experience, was possible about the real world, had only to point to geometry as an illustration. They said that no one but a madman would doubt its validity and no one but a fool would deny its objective reference.

On the other hand the empiricists of that time asserted that geometry had no certainty of a different kind from that of mechanics. They said that only the perpetual presence of spatial impressions made our experience of the truth of the axioms of geometry so wide as to seem absolute certainty. In saying this they went very much against the common sense of their day.

These two opposite views were held before the development of Non-Euclidean Geometry and they are still held by mathematicians and philosophers and by those who claim to be neither. If you join either group, you will have distinguished company.

Those who hold that the Euclidean axioms of geometry are intuitional dispose of Non-Euclidean Geometry by saying that the logical possibility of Non-Euclidean systems is irrelevant because the basis of geometry is not logic but intuition.

Those in the other group, who do not regard Euclidean Geometry as having absolute certainty, say that space as a whole may be Euclidean or it may be Non-Euclidean, and there is no means of determining which. You may say: "Why not measure the sum of the angles of a triangle and thus determine whether space is Euclidean or not?" Well, the measurement of the angles of the best approximation we can make to a geometric triangle will show an angle sum of approximately 180°. If the measurement does not show exactly 180°, the difference may well be attributed to the imperfection of measuring instruments. Therefore, the part of space in which the triangle lies is at least approximately Euclidean, and the use of Euclidean Geometry for practical purposes on this earth is justified. But the only triangles available for measurement are small as compared with the whole of space and it is characteristic of the two Non-Euclidean Geometries we have described that the angle sum is approximate 180° when the triangle is small.

So the question as to whether space is Euclidean or Non-Euclidean can not be determined by measuring the sum of the angles of a triangle. If space is Non-Euclidean, the angle sum of a triangle will not be 180°, but the difference may be too small to be detected by any means at our disposal. It has been suggested that a measurement be made of the angle sum of a triangle whose vertices are three fixed stars. Such a triangle ought to be large enough to furnish a definite result. But this measurement can not be made except by assuming that a ray of light is a straight line. But it is well known that a ray of light is not straight unless the medium through which it passes is homogeneous. So the result of the measurement, whatever it indicated, would be questioned and attributed to the bending of light rays by refraction.

Poincare, who was rated as one of the very greatest mathematicians in the world when he died a few years ago, maintained that the question whether Euclidean or Non-Euclidean Geom-

etry should be accepted was a matter of convenience and convention and not a matter of truth. He said that the axioms were definitions in disguise and that the choice between them was arbitrary.

A little while ago, I spoke of measuring the angles of the *best approximation we can make to a geometric triangle.* I had in mind the fact that there are no such things in physical nature as geometric points, lines and planes. The geometric entities of which we talk so freely are, I suppose, abstractions of things we do see in nature. There are very small particles of matter which approximate what we call a geometric point; there are very fine threads, which may be stretched so that we call them straight, which approximate geometric straight lines, or rather segments of geometric straight lines. The Geometric straight line is infinite. And there are flat smooth table tops whose thin top layers of varnish approximate pieces of geometric planes. In other words, these geometric points, lines, and planes are creations of the intellect with only approximate counterparts in nature. When one thinks of this fact, Poincare's remarks quoted above seem quite sensible, and one is inclined to replace the question "Which is true, Euclidean or Non-Euclidean Geometry?" by the question "Which is the more convenient, Euclidean or Non-Euclidean Geometry?" There is no doubt about the answer to this latter question. Euclidean Geometry is much more convenient than either the Lobatschewsky or the Rieman Non-Euclidean Geometry.

What then is the practical value of Non-Euclidean Geometry? Well, many a branch of mathematical science has been developed by mathematicians to satisfy intellectual curiosity with no idea that it would have any practical value. And later it has been found to have great practical value. This has happened so many times that it would be foolish for anyone to claim that a branch of mathematics which has no practical value today will never have any practical value.

In the Scientific American for the year 1920 there is a short article entitled "That parallel postulate" and there are other articles in that magazine for 1920 and 1921 about Non-Euclidean Geometry, especially that of Lobatschewsky. In them

you will find some of the things I have been talking about. Why do you suppose they are there? You may remember that, about that time, the newspapers gave much comment, but little light, on "Einstein's Theory of Relativity." The Scientific American offered a prize of $5000 for the best short essay on the subject of "Relativity" and the brief articles on Non-Euclidean geometry were put in to prepare the readers of the magazine for such statements as this, which were to appear later in the Scientific American, that the general theory of relativity leads to the conclusion that space may be finite though unbounded and therefore Non-Euclidean. As a matter of fact Non-Euclidean Geometry plays an important part in the literature of Einstein's "Theory of Relativity" as does also the geometry of four dimensions which to most people has seemed less practical and more weird and non-sensical than Lobatschewsky's Non-Euclidean Geometry. The pure mathematicians found a great deal of satisfaction in watching the physicists rush to learn what the mathematicians had done with Non-Euclidean Geometry when they, the physicists, discovered that some knowledge of that subject was necessary for the understanding of Einstein's work.

Having mentioned four dimensional space, I think it is time for me to stop before I am tempted to speak at length on that fascinating subject, although space of more dimensions than three is certainly "Non-Euclidean" in a sense and therefore what I might say on the subject would be appropriate to the title of this paper.

I close with a quotation from David Hilbert one of the greatest of modern mathematicians: "The most suggestive and notable achievement of the last century is the discovery of Non-Euclidean Geometry."

THE STUDY OF MATHEMATICS UNDER THE INDIVIDUAL SYSTEM

By MARY M. REESE
Skokie School, Winnetka, Ill.

About three years ago the system of individual instruction was introduced in the Winnetka schools by our superintendent, Dr. Carleton Washburne. This naturally had to be done gradually for after establishing definite goals for the work which must be accomplished in each grade, the material had to be prepared very carefully so that it would be as nearly self instructive as possible and the children could use it with a minimum amount of help from the teacher. Under this system a child progresses at his own rate of speed, neither being held back by slower pupils nor forced to go forward too rapidly for thorough understanding.

There is no subject, I believe, where there is such a difference in ability among pupils as in mathematics. Under the class instruction, many times children have reached the intermediate school with inadequate foundation in fundamentals because they were slow to grasp at least some process, but had to progress with the class.

Under the individual system a child cannot fail. He never repeats a grade although he may take more than a year to cover the work of that grade but the next year he commences where he left off. This time may be made up later if he has a good foundation. On the other hand many children are able to accomplish the year's work in less than the given time and if so are promoted to the next grade's work in that subject at once but they never skip a grade. A child does not change rooms each time he is promoted, the groups being changed usually once a year.

The arithmetic in all grades contains much practice work with answers so that the child can test himself. The preparation of these practice books has naturally been a big task. This has been done by all the teachers who are to use the books, their work being mimeographed and assembled into books which are given to the pupils as they need them. The work had to be taken up step by step with a quantity of practice work for each step. Each book is provided with answer sheets so that each child can correct his own work. Each lesson is followed by test corresponding with the work taken up in the lesson.

In the lower grades some of the development must be done orally but most of this work has been planned so that the children can understand it with a minimum amount of oral instruction. In the upper grades all of the work is developed in their practice books. If this is not understood by a pupil he has an opportunity to ask for further assistance. When he feels he is ready, he takes a test on the new step. This test is corrected by the teacher.

In the intermediate school, if he makes any error in his test, he corrects mistakes, takes more practice work if his test shows need of it, and takes a second test, and if necessary, a third test.

There is necessarily some bookkeeping in connection with this work so that a teacher may know at any time where each pupil stands but usually only a record is kept of correct tests.

On the other hand all the practice work is corrected by the children themselves and not every child hands in tests every day, making fewer papers for the teacher to correct than under the class system. At the end of the month a goal book is sent home for parents inspection and signature. This book contains a list of all the goals which must be reached for that grade, and shows how far the child has progressed toward the achievement of these goals. If he is promoted in any subject he is given the goal book for the next higher grade.

The arithmetic pages of the seventh and eighth grades are given below. On the fundamental page, "speed" means the number of examples of standard difficulty worked in three minutes; "accuracy" is the percent correct.

SEVENTH GRADE, ARITHMETIC FUNDAMENTALS—GOALS

Addition Review	Sp. 6.	Acc. 100 per cent		
Subtraction Review	Sp. 12	Acc. 100 per cent		
Simple Multiplication	Sp. 3	Acc. 100 per cent		
Compound Multiplication	Sp. 2	Acc. 100 per cent		
Long Division	Sp. 2	Acc. 100 per cent		
Fractions	Sp. 4	Acc. 100 per cent		
Decimals	Sp. 4	Acc. 100 per cent		

Course Begun ____________________ 192____.

Promoted to Grade 8 Arithmetic Fundamentals ____________ 192____,

____

Teacher.

SEVENTH GRADE ARITHMETIC GOALS

EXPLANATION:

In the Seventh Grade, Parts I and II are essential for promotion to Grade 8 in Arithmetic. This is in addition to work in fundamentals, the goals of which are given on the preceding page. The advanced part is to be completed as far as possible.

	Oct.	Nov.	Dec.	Jan.	Feb.	Mch.	Apr.	May	June
Self-Reliance									
Diligence									
Deportment									

In the seventh grade, the required work is divided into two parts; the first includes application of percentage such as profit and loss, discount, commission and interest, and the second part is work in mensuration. In this section of required work are the subjects in mensuration which are essential to every child, even if they leave school at the minimum age.

In addition to the required work mentioned above, a general review of all fundamental operations is required. To receive an O. K. in any operation the pupil has to work accurately a given number of problems in three minutes. If he makes any error, he has to take additional practice work and tests until the required goal is obtained.

The individual work covers the minimum amount of work which must be accomplished for promotion to a higher grade but there is much social work done in all the grades. This work supplies the life and the co-operation among the pupils that is lacking to a certain extent in their individual work. As arithmetic is more of a drill subject and more easily adapted to individual work those other subjects, much of the social work is found in other studies. From one-third to one-half of the pupils' time is occupied by this social work which is an important part of their curriculum; although their marks and promotions are based entirely on their individual work.

In the eighth grade, however, the work, although outlined for individual work and followed by tests on the essential points for which each one is held responsible is mostly developed by dis-

cussions or by some form of social work. The required work in this grade is composed mostly of business subjects. The topics themselves are more important than the examples usually presented in connection with them. How many of us ever learned very much about insurance, taxes, stocks and bonds from the numerous examples we worked? It is our aim in the eighth grade so to vitalize these subjects that they have a real meaning to the boys and girls. Not that they will have a comprehensive

SEVENTH GRADE ARITHMETIC GOALS

Part I—Percentage Application	
1. Percentage Review ______	______
2. Profit and Loss	
a. Finding Gain: Cost and rate given—Test 1______	______
Test 2______	______
b. Finding Loss: Cost and rate given—Test 3______	______
c. Finding per cent of Profit or loss—Test 4______	______
Test 5______	______
3. Commission ______	______
4. Discounts ______	______
5. Interest and Amt. for years—Test 8 ______	______
6. Interest and Amount for yr. and mo.—Test 9______	______
Review Test ______	______
Part II—Mensuration	
1. Review of Long Measure—Test 1 ______	______
2. Square Measure—An acre of land—Test 2 ______	______
3. Volume of Rectangular Prism	
Test 3 ______	______
Test 4 ______	______
Test 5 ______	______

4. Circles	
Comparison of Diameter, Radius and Circumference—Test 6______	______
Test 7______	______
Test 8______	______
Test 9______	______
Test 10______	______
Review Test______	______
Completed ______, 192____.	
ADVANCED	
Area of Triangles—Test 1______	______
Area of Circles—Test 2______	______
Volume of Cylinders—Test 3______	______
Short Cuts in Multiplication	
Test 4______	______
Test 5______	______
Test 6______	______
Short Cuts in Division	
Test 7______	______
Test 8______	______

Course Begun ______192______.

Promoted to Grade 8 Arithmetic______192______, ______
Teacher

knowledge of business, but they will get enough to lay a foundation for more as the needs arise and to have an understanding of business terms which they will hear and of which they will read.

The children are taught banking by running an imaginary bank. They all receive check books which are printed in the school print shop by the boys and, following directions found in their practice books, they make out checks to each other and fill

EIGHTH GRADE ARITHMETIC GOALS

I. BUSINESS FORMS

1. Bills ....
2. Receipts ....
3. Cash Accounts ....

Banking

1. Checking Accounts ....
2. Savings Accounts ....

Real Estate

1. Buying and Leasing ....
2. Mortgages and Notes ....
3. Bank Discount ....

Insurance

1. Fire Insurance ....
2. Life Insurance ....

Taxes

1. Local Taxes ....
2. National Taxes ....

Stocks and Bonds

1. Organization of corporation ....
2. Stock ....
3. Bonds ....

Review—Factoring ....

II. ADVANCED

1. Square Root — Test 1 ....
   Test 2 ....
   Test 3 ....
   Test 4 ....
   Test 5 ....
2. Hypotenuse of Right Triangles ....
3. Use of the Formula—
   Test 1 ....
   Test 2 ....
   Test 3 ....
4. Use of the Equation—
   Test 1 ....
   Test 2 ....
   Test 3 ....
   Test 4 ....

Date Begun .... 192....

Work Completed .... 192....

.... Teacher

out the stubs in their books. The checks received are endorsed according to directions and with deposit slips made out are deposited in the bank, the children taking turns as tellers. If any error is made the check has to be rewritten. After all the required checks are written correctly and deposited, they balance their accounts and take a test which covers the essentials. If any mistake is made they take a second test.

They have studied stocks and bonds by forming a stock company. They appoint incorporators who, after deciding on the amount of capital stock and the cost a share, solicit stock from other members of the class, which is paid for by check. Meetings are held at which directors and officers are elected and the subject studied learning the meaning of preferred and common stock and other important terms used. At the end of the first imaginery year, dividends are declared, the children receiving their dividends in checks made out by treasurer. The corporation then borrowed money by issuing bonds.

This year the plan was changed somewhat. A real corporation was formed with capital stock amounting to $300, the incorporators selling shares of preferred stock yielding 7% interest at 10 cents a share, in lots of from one to fifty shares, to not only the eighth grade but to lower grades, parents and teachers. Up to date about $250 has been paid into the treasury by subscribers who have received regular certificates with the seal of the corporation. At the end of the first month, business being prosperous a quarterly dividend of 2% was declared, each stock holder holding five or more shares receiving his pennies from the treasurer.

This company, known as the Skokie Finance Corporation has under its control the school paper which is edited and printed by the boys in the school shop. It also runs a very active school store which sells school supplies to the pupils. It expects to control other enterprises that arise. All bills are paid by checks made out by the treasurer and countersigned by either manual training or mathematics teacher.

The store is operated by children who are so far advanced in their work that they have extra time. These children are taught simple accounting and are keeping a set of books in connection with this work.

Next year we hope to continue this work, issuing bonds and enlarging its usefulness.

In addition to the essentials which must be achieved by the child, in both the seventh and eighth grades an advanced course in mathematics is prepared for those who finish the essential part of the work before the close of the year. This is entirely individual and the child works as far as he individually is able. As the essential part is expected of the slowest, the majority will do some of the advanced work and many finish all of it. As the eighth grade advanced work is designed as a help to high school mathematics, the children are urged to work as much as possible of this if they are up to standard in their other studies.

The attempt has been made to fit the work to the needs of the children and not fit the children to the course prepared.

PROBLEMS CONCERNING THE TEACHING OF SECONDARY MATHEMATICS[1]

By ALFRED DAVIS
Soldan High School, St. Louis, Mo.

It is not our purpose to enumerate all the problems connected with the teaching of secondary mathematics, even if that were possible, much less to attempt to solve all of them. It is, of course, much easier to state an educational problem than it is to solve it. A prominent educator said, not very long ago, that this is an age of diagnosis in the field of education, and we have not yet reached the stage of prescribing adequate remedies for our ills. We shall survey briefly only a few outstanding problems, and we hope to suggest possible solutions for some of these.

It is not very long since it was assumed by some who were in high places educationally, that the long accepted idea that a general disciplinary value could be attributed to a study, was unwarranted. That is, the study of a subject like mathematics could be valuable only in the further study of mathematics or in its application to other fields. Simultaneously with this the teacher of mathematics was confronted with the question, "Why do you teach mathematics to high school students, and what do they gain from it?" For the moment this was staggering. The attempt to show that the subject trained the mind was treated as ridiculous, or else it was met with the statement that several other subjects could do this as well, or better. The practical usefulness of mathematics did not answer the question since some were ready to show that the average student would need little on this account. Unless a satisfactory answer could be readily given to the question mathematics could not long continue as a required study in our high schools.

It has been shown more recently that the above assumption is not entirely correct, to say the least. No one of real prominence in education will today wholly deny the transfer of training. However, this transfer is not conceded to be automatic, and is limited to a correspondence of elements in the various fields. In

[1] Read before the Missouri Section of The Mathematical Association of America at Kansas City, Mo., November 18, 1922.

this connection it is interesting to note the following from the "Revision of the Course of Study in Arithmetic" for Chicago and dated September 1921:

"Its mental discipline—the power it gives the mind in reasoning, in habits of application, and in exactness of statement, the power to think closely, concisely, and accurately has a fundamental importance in attaining success in the various and progressive walks of life, and the teaching and study of arithmetic contribute a large and peculiar share of that success. While most of the mental discipline of arithmetic may be secured from those portions that may be called practical, the importance of right logical thinking that underlies and gives the highest excellence to arithmetical calculations must not be underestimated. The logical value of arithmetic lies in training the mind to interpret correctly the problems, to think the solutions of problems apart from their computation, and to cultivate the power of sustained attention and reasoning." Indeed, this might be taken as an argument for the study of mathematics in high school. However, that there is still a problem in connection with this matter may be seen in the increasing tendency of many of our large and important high schools to graduate students who have not studied mathematics, and in the resulting difficulties of the graduates. The matter is further emphasized by such action as that recently taken by the state of Ohio, to the effect that no high school in the state need require mathematics of its students. Commenting on this the Mathematics Teacher says:

"The effect will probably be to improve the 'tone' of the average mathematics class . . . . . The worth of mathematics as a mental discipline is too firmly established to require argument at this late date"

California seems to have taken action similar to that of Ohio with the result that opportunity is offered at the State University for the study of elementary algebra and geometry.

Much might be said of the contributions of the study of mathematics to intelligence and culture, and of the dependence of the progress of civilization upon it, but it is not our purpose to enter into these fields; suffice it then to say that much remains to be done in defining the objectives for high school mathematics and in giving these publicity.

One of the outstanding reasons for focusing attention and criticism on mathematics in recent years is the large number of failures among students taking the subject. It has been quite common to have as high as a third of a class fail, sometimes fifty per cent., while many students fail repeatedly. It is quite natural to raise the question as to why a subject should be required of persons who seem unable to master it. The question becomes an economic one. The four year course is likely to be extended a term or more with a proportionate increase in expense to the community. It is likely that there will be a corresponding return to society for this extra outlay? It must be admitted that the return is likely to be in inverse proportion. The responsibility for this intolerable situation has been charged to the subject. Now this is probably a mistake. The difficulty should be met by a classification of pupils according to intelligence and ability; by improvement in the arrangement and in the presentation of subject matter; and by selecting as teachers those who are specialists in subject matter and in the technique of its presentation. Notwithstanding the many efforts recently made to adjust these matters they are still serious problems challenging the teacher and the administrator. Until they are more completely solved we may expect dissatisfaction and impatience as illustrated in the recent action of the State of Ohio. Let us consider these problems in some detail.

The ordinary high school class in mathematics may be considered as consisting of three groups, five or six bright pupils, about the same number of slow or dull pupils, with the majority of average ability and ranging between the two extremes. Such a class should be divided into three parts for purposes of effective instruction. This, however, is a problem for the administrator rather than the teacher, and it is not easy of solution. At the present time we teach the average pupil. The inferior group is probably entirely included in the failures, while the superior group marks time. A few years ago high school students were a more select group and the classes fairly uniform, but with the rapid increase in numbers we have a wider range in ability. With further increase in numbers the difficulty is likely to increase. Professor George S. Counts of Yale Uni-

versity, has made a study of the high school population of four cities; Bridgeport, Conn.; Mount Vernon, N. Y.; St. Louis, Mo.; and Seattle, Wash. Considering these as representative of the entire country, he concludes that only 25%, or 2,000,000, of those of high school age are enrolled in high school at present, and that these are largely a select group from the more fortunate social and economic classes, while the lower classes of labor have as yet hardly begun to think in terms of secondary education. The high school idea is bound to spread. This is indicated in the fact that large high school buildings are no sooner erected than they are filled to overflowing. As we continue to draw on the 75% not now enrolled the need for better classification is bound to increase.

The junior high school is aiding in the solution of the problem, and I suppose the Ben Blewett school of St. Louis may be taken as illustrative of what is being attempted. When pupils enter this school from the sixth grade they are given the Otis intelligence test. On the basis of the resulting intelligence quotients they are divided into A, B, and C groups. For example, the class which entered last February had about one-fifth A's, with I. Q's above 120; about the same number in the C group with I. Q's below 90. while those between 90 and 120 formed the B group. These adjustments are usually revised after about five weeks when the actual achievement of the pupils in the class room has been observed. The chief difference in these groups is in the length of time required for graduation. The A's require two years, the B's two and one-half years, and the C's three years. If a pupil is over age for his grade he is placed in a "rapid promotion" group under a strong teacher, taught merely the necessary elements of the subjects, and then passed on to a grade of corresponding age. The pupils are marked according to what might be reasonably expected of the group. Accordingly, a high mark in a C group would not be a high mark for a member of the A group.

Approximately the same courses in mathematics are required of all pupils in grades seven and eight. The usual arithmetic required of these grades is covered, supplemented by some intuitional geometry, and including the use of the equation. Mathe-

matics is elective in the ninth grade except for those who take the scientific course or the manual training course. The C pupils who take manual training are given a special course in shop mathematics, which is not given credit at Soldan Senior High School. All others who take mathematics are given general mathematics [1] with the emphasis placed on algebra.

On entering Soldan the A's are assumed to be ready for geometry. The B's are credited with one-half year of mathematics and begin the second half-year of algebra, while the C's are given no credit in mathematics and so begin algebra, in case they elect mathematics.

Almost all the Blewett graduates enter Soldan. However, the two districts are not co-extensive, and so Soldan receives probably as many from the eighth grade as from Blewett. Soldan was built to accommodate 1800, consequently, with a student body of more than 2500, it is crowded. These conditions interfere with maintaining the A, B, and C groups found at Blewett. At Soldan many pupils are obliged to take mathematics to meet college entrance requirements, while those in the manual training and the scientific courses must take it, the former at least two years beyond the eighth grade, and the latter four years. The result is that the junior high graduates are thrown together with others who have had the traditional courses in the ninth grade and confusion follows. The C pupils are lost and are largely listed as failures. Some attempt has been made to run parallel classes so that the weak pupils might be separated from the others and allowed to move more slowly, taking a longer time to complete the same work as the others. This has not been successful, one reason for this is the difficulty of keeping these pupils separated for more than one term. Obviously, unless the pupils take mathematics under compulsion of some sort the number electing it under these conditions is destined to decrease. For the first time all the high schools are offering general mathematics. It is the hope to place all the weak pupils who elect mathematics in these classes. A year of geometry may follow with credit but the pupil will not be considered prepared to continue beyond this without the usual algebra. This is an

[1] Beginning with the present term algebra is offered as a ninth grade elective at the junior high school.

experiment, and one of the greatest difficulties is the lack of suitable texts. The proper classification of pupils and properly differentiated methods of treatment are problems awaiting successful solution. The situation might be met with specialized technical high schools so operated as to meet special aptitudes of pupils. Again smaller classes, with less crowded conditions, and an increased number of teachers would help. These adjustments would involve additional expense making the problem an economic one and consequently slow of solution. It may be that we should eliminate some students at the completion of the eighth grade because of limited ability. However, it is important that these problems be met vigorously and open-mindedly until a successful solution is worked out.

Again, the courses in mathematics have become static. They were developed under conditions which have long since changed. The National Committee on Mathematical Requirements has labored to reorganize them, and its final report just about to appear will be a valuable contribution towards that end. Special schools have attacked the problem. And teachers here and there in public schools have made some contributions quite worth while. But progress is extremely slow. The experienced teacher in the class room reminds one of the bishop who, in the course of his travels, preached the same sermon one hundred and one times. When asked why he did this he replied by asking the question, "If you were going on a hunting trip and had the choice of two guns, one of which you had already tried repeatedly and had always gotten the game, while the other was new and untried, which would you choose?" It is not easy for teachers to avoid ruts in their teaching. It is probable that tried methods should be changed slowly and cautiously. The same may be said of subject matter. The junior high school has given impetus to the movement for improved courses, but even here it has recently been said as a criticism that while the administrative side has advanced rapidly, the courses have changed but little.

One of the important reasons for the junior high school is to avoid, if possible, the marking of time in the seventh and eighth grades. To realize this in mathematics the subject should

probably be required throughout these grades and it should include work beyond the traditional arithmetic. The subject is frequently elective in the ninth grade, but whether elective or required, it should at this stage include more than elementary algebra. It seems desirable to include both algebra and geometry, with probably an introduction to demonstrative geometry, and possibly the use of the sine, cosine, and tangent functions of an angle. By such procedure a second great purpose of the junior high school may be promoted, namely, testing the aptitude of pupils for a study. The work in the junior high school should be a unit. That is, it should not depend entirely on further work in the subject to have meaning and value. This is especially important for the pupil who leaves school at the end of the ninth grade. The work should at the same time be so arranged that the pupil will be prepared to continue his work in the senior high school without interruption or embarrassment.

Senior courses need reorganizing in such manner as to insure a maximum of good results from previous work. This problem has scarcely been thought of as yet, but it must receive attention. It is emphasized in the fact that mathematics courses in the senior high school are likely to become and remain elective. If we continue to try to fit the new type of pupil into the traditional courses which have had little or no modification, the subject will become unpopular and few will elect it. This result is apparent in many parts of the country today. The problem is not easy to solve. It requires the combined efforts of numbers of teachers in class room work, and the careful checking of results. Some attempt along these lines have been made by the Horace Mann and the Lincoln schools of New York City. The following may be suggestive.

The work in algebra may be condensed so that not more than an additional year may be required to complete it. Probably too much time is spent on solid geometry. This might be combined with plane geometry and the whole condensed so that one year would complete it. Or the two geometries might be kept separate but so organized that only one year need be devoted to geometry. In this arrangement the pupil could get the train-

ing that the study of geometry would give, and he could certainly gain the information he needs from the subject. This leaves a year for the mathematically inclined pupil to devote to advanced studies. A choice of courses may be offered for this year, any two of which may be taken, in succession. Trigonometry with applications might take a half year. This may be followed by surveying, although this subject is probably better suited to the technical school. Advanced practical mathematics adapted to shop work, elementary statistics, or a brief course in the history of mathematics might be offered. However, the most fascinating fields from which to draw for an advanced senior course are analytic geometry and the calculus. Much is said about applying mathematics to practical situations, and the finding of real problems. Our high school courses in mathematics are sometimes rightly criticised as barren because they either follow the wrong trail or they stop too soon. No one can have an adequate idea of the meaning of mathematics and of its beauty and power in solving every day problems until he has studied at least the elements of analytics and the calculus. Many of us who teach upper class students tell them of the wonders of these subjects and urge them to plan to study them in college. This suggestion, however is tantalizing to the student who never goes to college, and this seems to be an increasing group. President Coffman, of the University of Minnesota, concludes from a recent study that as far as Minnesota is concerned not more than 25% of the graduates of high schools, during the next ten or twenty years, will enter college. This proportion will probably be greater as we continue to draw on the 75% of high school age who are not now in high schools. Since so many who could understand the fundamentals of the calculus will never know what it means unless they study the subject in high school, and since further progress in mathematics and its application to the needs of the world are dependent on the use of the calculus, it is a misfortune to deprive mathematically inclined pupils who could continue the study after leaving school, of this key to a world unknown or mysterious to the many, yet one of the most powerful tools of human thought. Many topics now included in algebra and solid geometry are but weakly defended as compared with analytic geometry and the calculus. Problems re-

lating to the curved paths of moving bodies, areas of surfaces bounded by curved lines, the almost magic ease with which maxima and minima values can be obtained when equations are written from known conditions, and many problems of everyday occurrence open a new world of interest. We may tell the pupil that a parcel post package having a combined length and girth of 84 inches will have a maximum volume when the girth is twice the length and when cylindrical, but he will surely be interested in finding this out for himself. We may tell him that the arch of a bridge will require least material when circular, but that to have the greatest strength it must be parabolic, but he will want to know why. He will be interested to know why automobile lights, and the mirrors of reflecting telescopes must be paraboloidal in form; why a shute for a bathing pool or for carrying coal should be the form of cycloid; why an ellipsoidal room is a whispering gallery; how deep to cut the corners of a rectangular piece of tin to make a box of maximum capacity; how to find quickly and easily, in most cases, the turning points of a curve from an equation when it represents a power of a variable above the first. He will, for the first time, have some adequate appreciation of the fundamental laws of the universe, and he will be able to use them to interpret observed phenomena. The great engineering works of civilization will have new meaning for him. The serious importance of it all will be driven home on learning of such a calamity as that of the Quebec bridge a few years ago. Hundreds of lives and about $7,000,000 worth of property were destroyed, because, we are told, an engineer was so poorly informed in his mathematics that he supposed a beam twice as large would be twice as strong. Sir Isaac Newton, the great intellect of the ages, invented the calculus to apply his law of gravitation to the motions of the heavenly bodies. It is, however, such a powerful instrument of investigation, and of interpretation of natural phenomena, that it has become fundamental to our civilization. Little has been done so far to adapt the elements of these two great subjects, analytics and the calculus, to the learning of high school pupils. Nevertheless, it can be done. It is a problem for the future, but its solution should not be long delayed.

Superintendent John J. Maddox, of St. Louis, said recently in addressing a group of high school teachers that the fundamental idea on which the curriculum is to be built is not that it is simply to impart knowledge, it is not simply to learn how to earn a living, and it is not merely to give culture; but the idea is to affect the behavior of folk. The controls of conduct are: establishing habits, imparting knowledge as to what to do and how to act, and the building up of great ideals. Every study, to find and to hold a place on the curriculum, must be tested in the light of these great ideals. The revision that we would suggest for courses in mathematics must be in accord with these great purposes.

Time will not suffice to enter completely into the teacher problem. The teacher is of first importance in any plan of education. Buildings and equipment may be the most modern, classification of pupils may be perfect, text books and courses of study may be ideal, but without the teacher these are like the engine without steam, like the electric light fixtures without the electricity. It is the personality, the enthusiasm, and the power of the teacher that count. He must have the ability to impart the qualities and the ideals which he himself possesses to accomplish what we would call education.

While breadth of culture is essential for the teacher he must be a specialist in the subject he attempts to teach. May the day soon pass when the specialist in English, or in history, or in language is assigned classes to whom he is to attempt to teach mathematics. It is not usually assumed that a college professor can teach a number of subjects with equal facility and effectiveness, neither can this be reasonably assumed of the high school teacher. We were asked not long since, by men prominent in the college of education in one of our large universities, "How much mathematics should a prospective high school teacher of the subject study?" When we answered that he should at least know the calculus, they expressed amazement. They asked, "How can that be? He is not going to teach the calculus." It seems trite to say that a teacher should know more of a subject than he expects to teach, that this should be true is merely a matter of common sense. If the revised courses we have suggested are to be effective we ought to insist that the teacher should have followed some branch of mathematics be-

yond the calculus, and he should continue to be a student of advanced work in the subject.

The teacher should be trained in the teaching of the subject. A few years ago the writer visited a geometry class in one of our large city high schools, in which all the pupils had failed at least once in the subject. The teacher was insisting that the pupils learn verbatim the words of the text. Worse yet, the pupils were required to memorize the *numbers* accompanying the paragraphs. I asked the purpose of this and was told that in this way the pupil could know whether the reason given for a statement preceded that point in the work. One can imagine the effect of this upon a group who had previously lost interest. Too many teachers are attempting to teach without training in the art of teaching, and without knowing the fundamentals of educational psychology.

Since mathematics is to be an elective in most of our high schools, the teacher must "sell" the subject and the idea of its importance to the community he serves, both by good teaching in the class room and by being a source of information on the subject outside the school. This requires that he be, not only a constant student of the subject, but also a student of education in general. He should learn from other teachers both by reading and by personal contact with them. He should learn from the world about him. And he should know the educational problems of the day, and if possible contribute something to their solution. One of the most important problems is how to secure well equipped, enthusiastic, sympathetic teachers.

We have suggested a few of the outstanding problems connected with the teaching of secondary mathematics. Some of these are partially solved but much remains to be done. Their more complete solution will aid much in giving mathematics its proper standing both in and out of the class room. The problems are urgent since the subject is becoming increasingly important as an aid to the progress of many other branches of science, and its study has an educational value that cannot be treated lightly. With better classification of pupils, readjusted subject matter, and with teachers who are themselves students, and who can teach others how to study, the problems will be largely solved. However, true solutions must come largely from the teachers of mathematics themselves.

THE FUTURE DEVELOPMENT OF MATHEMATICAL EDUCATION[1]

By PROFESSOR CHARLES N. MOORE
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You have already heard this afternoon of the work of two organizations that have been and are actively interested in the improvement of mathematical education. You have also had presented to you programs for adapting mathematical instruction to the needs of two rather recent types of school organization. It is apparent from this afternoon's discussion alone that the teachers of mathematics here and throughout the country are alive to their opportunities and their responsibilities. They realize the great service to society which they can perform by selecting from the vast store of mathematical knowledge those elementary methods and processes that are of widest use in the modern world, organizing them into coherent courses, and presenting them effectively to their classes. It is apparent to the careful observer that existing mathematical courses have not been constructed with due regard to the relative importance for the general student of the different mathematical methods and principles that are available for instruction in school and college. Our courses have been arranged primarily for the benefit of those who will continue their mathematical education. That they do contain much material of great use to the general student arises from the fortunate circumstances that most of the processes in elementary mathematics have some important applications in the world of to-day. But the conscientious teacher of mathematics will not allow his good fortune in this respect to paralyze his initiative. If we can add considerably to the usefulness of our courses by reorganizing them, by all means let us do it. It is certainly our duty and it should be our pleasure.

If we consider the various individuals who in the past have contributed to the world's progress, we find that we can group them into three principal classes. We have first the faithful plodder who sees small improvements that can be made here and

[1] Delivered before the Mathematics Section of the Missouri State Teachers' Association at St. Louis, Missouri, November 3, 1921.

there and contributes his best energies to making them. We owe much to the combined efforts of the plodders and we should have proper appreciation for them. Their chief fault is that at times through their lack of vision they sacrifice the greater good to the lesser. The next group is composed of the dreamers or idealists. They are animated by a great vision of things as they ought to be, but they are blind to practical difficulties, and for this reason they rarely succeed in bringing about by their own efforts the reforms they advocate. But they do serve to galvanize the indifferent by their personal enthusiasm, and in this way they frequently make it possible for other men to carry through to success the tasks in which they have failed. In spite of their limitations, let us render due honor to the idealists. The third group comprises those persons who combine the qualities of the idealists and the plodders. They have the fervent enthusiasm which comes from the vision of a great ideal, but they have also the practical common sense to realize that we may never reach that ideal and that at best we must proceed toward it by steps that are slow but sure. This is the most useful group of the three, and we should all strive to belong to it. We should have high ideals, because they furnish enthusiasm in our work and guidance of our efforts, but we should also have the patience and courage to work our way toward them through the maze of practical difficulties in which we inevitably find ourselves.

Since teachers of mathematics are at present so generally occupied with the problem of improving mathematical education, and since furthermore teachers are more apt to be idealists than the generality of mankind, it seems fair to conclude that many teachers of mathematics have more or less definite ideals as to what mathematical education ought to be. Granting that this is the case, it would seem desirable that all of us who have such ideals should endeavor to formulate them so that by comparison and discussion we may come to approximate agreement as to our larger aims. With this idea in mind, I am going to present briefly my own views as to the ideals which should govern the future development of mathematical education.

We who have been long engaged in the study and teaching of mathematics realize full well that it is one of the most essential elements in the warp and woof of our modern civilization. We know that most of our commerce, industry, and science depends to a greater or less extent on mathematical knowledge, and that many of the arts owe a considerable debt to this fundamental subject. We are quite aware that the complete wiping out of every vestige of mathematical knowledge would immediately paralyze the activities of the civilized world. But to how many of our students, even the more gifted ones, do we bring home this realization? To how many do we give just a glimpse of the amazing vitality and power of a living and growing mathematical science?

In every great commercial activity of the modern world much attention is paid to the sales force. It is not enough to produce a worthy article; you must persuade the buying public that it is a worthy article. I believe that we mathematicians and teachers of mathematics have in the past been woefully negligent of the selling end of our activities. We have produced mathematical knowledge in enormous quantities; such of it as the rest of the world has been wise enough to utilize has been exceedingly beneficial. We have handed this knowledge over the counter very cheerfully to such students and inquirers as came to seek for it. But in general we have made no systematic and intensive effort to persuade our students and fellow citizens of the value of mathematical knowledge. By selling activities in connection with mathematics and the teaching of mathematics I of course do not mean selling in the ordinary commercial sense. What we must aim to secure for mathematics is not mere dollars and cents, but an appreciation of its value and an enthusiasm in its pursuit. Adequate material support will follow as a natural consequence.

The important question then is what shall we do to make people in general and particularly our students realize the value of mathematical education? We must of course organize our courses so as to include as many as possible of those elements of mathematical knowledge that are of real importance in the world of today. We must eliminate from elementary courses

difficulties that are purely artificial and which do not arise in the ordinary applications of mathematics. There are plenty of real difficulties to include in our courses; we can well dispense with the artificial ones.

Such topics as we do decide to teach on the basis of the above principles should be presented in close connection with some of their important applications. The student who is learning mathematics should be made to realize that he is acquiring knowledge that real live people find it highly necessary to use in human activities of fundamental importance.

All this is easy to say. It is by no means easy to do. As I told you at the beginning, I am describing an ideal, a vision. In choosing applications to reinforce the teaching of mathematics, one must proceed with great care. The application may be so technical as to be more difficult to understand than the mathematical principle on which it is based. We cannot at the same time study mathematics and a multitude of other difficult subjects. It should be the business of all earnest teachers of mathematics to keep a constant lookout for simple applications of elementary mathematics that are of importance in the modern world and which at the same time can be readily comprehended by the average student. Some of the more technical applications may be referred to and described in a general way, but the applications that drive home the principle should be as simple as possible.

In addition to bringing home to the student the wide use of mathematical knowledge in the activities of the modern world, we must also give him some notion of its origin and growth and its important role in the development of our civilization. We must not let him rest under the impression that mathematics was invented in order to provide intricate and vexatious puzzles for the adolescent mind. We must demonstrate to him that man was led to the pursuit of mathematical knowledge by his eager desire to understand the universe and to control the forces of nature, that he found this knowledge essential for the higher developments of trade and commerce and all of the other varied developments that have had a place in the creation of

our present day civilization, in short that the progress of the world is now and always has been bound up with the development of our knowledge of mathematics.

We must draw from biography as well as history in our endeavors to create in our students a real enthusiasm for mathematics. I believe the average young American is almost if not quite as ready to applaud mental achievement of a high order as he is to applaud prowess in the world of sport, provided only that the former is properly presented to him. I do not see why the tenacity of purpose and the prodigious mental exertion by means of which the great heroes of our science scaled hitherto impregnable heights of mathematical theory should not arouse something of the same thrill as the description of the physical endurance and courage displayed in the ascent of the Matterhorn or the dash to the North Pole. Let us make plain to our disciples that the quest for mathematical knowledge is one of the most important and most fascinating portions of the great adventure, of man's eternal effort to penetrate further into unknown regions and master them for his possession and use, and they will be convinced that while mathematics may be a difficult subject it never can be a dry subject.

I know this is a large program that I have here mapped out, but I think it is an entirely possible program if we adopt it for our own and put forth our best efforts to realize it. These efforts should of course include much self-development. He who would create enthusiasm in others must first have enthusiasm of his own, and it is difficult to remain enthusiastic about a subject in which we are not steadily growing. A complete knowledge of all the mathematics thus far discovered is not possible, even for the most gifted mathematician of the age, so there is nothing to limit our advance in the science, even though we are not ourselves engaged in investigation. I consider it one of the most important duties of every teacher of mathematics to be constantly increasing his stock of mathematical knowledge. He should in particular study the history of his subject and its intimate relationship with the history of other sciences, and the progress of civilization. He should be familiar with the names and the principal achievements of the great leaders of mathematical thought, past and present.

Finally the teacher of mathematics should never allow himself to become static in regard to his work. No matter how much we may improve the teaching of mathematics, something will always remain to be done. We who deal so constantly with variable quantities have no license to forget that the world about us is in an eternal state of flux and that we must ever adjust ourselves to changing conditions. In our work just as in all other of the world's activities, it is essential to remember that

> "New occasions teach new duties; time makes ancient good uncouth;
> He must upward still and onward who would keep abreast with truth."

THE FUNCTION CONCEPT IN HIGH SCHOOL MATHEMATICS[1]

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The Fundamental Character of Functionality. The perimeter of a square depends on the length of a side. The area of a circle depends on its radius. The time of vibration of a simple pendulum is related in a very definite way to the length of the pendulum. The distance passed over by a body moving with uniform velocity depends on the velocity and the time. The amount of a sum of money placed at compound interest depends on the principal, rate, and time. The size of the crops depend on the acreage, on heat, moisture, fertility of the soil, and the industry of the farmer. The rent one pays for his house depends on the size, improvements, location, inflation of the currency, and the conscience of the owner. It would be possible to enumerate thousands of instances in which things are related to things. The notion of relationship or dependence is one of the most, if not the most, fundamental notion of mathematics. It is usually designated as the principle of functionality. The notion of function has long been consciously recognized by mathematicians. Leibnitz used the word function to designate different kinds of geometrical magnitudes associated with a variable point on a curve. The meaning of the word has been broadened from time to time so that at present it implies relationships in which one object corresponds in some definite way to another object. Thus if we should choose x to be a variable representing any one of the hours today and y another variable representing the temperature, then we could set up a correspondence between the values of x and y in such a way that for any value of x there is a unique value of y. Under such restrictions y is said to be a function of x.

The Function Concept as an Organizing Principle. If the function concept is the most fundamental thing in mathematics, why should it not be used as an organizing principle? This question seems to have occurred first to Professor Felix Klein

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[1] Read at the Chicago Meeting of the National Council of Mathematics Teachers, March 1, 1922.

and was expressed by him in a paper read before the International Congress of Mathematicians which met in Chicago in 1893.

Professor E. H. Moore, in an article on "The Cross-section Paper as a Mathematical Instrument" appearing in The School Review, May 1906, says: "In this note I wish to suggest possibilities, which may have escaped attention, in the systematic use of cross-section paper as a unifying element in mathematics. I know of no medium serving to bring together so easily the three phases, or dialects, of pure mathematics—*number, form, formula*—and to lead so directly to the concept of functionality —a concept which, since the seventeenth century, has dominated advanced mathematics and the sciences, a concept which in the twentieth century, according to the auspices, will play a fundamental roll in the reorganization of elementary mathematical education. Students will gain a more easy and perfect mastery of mathematics, and their work will be full of richer direct and indirect value for them, when the primary emphasis is laid on the recognition, the depiction, and the closer study of functional relations between variable quantities."

Since the appearance of these papers a number of books on first year college mathematics organized about the principle of functionality have been written in this country. It is only recently that such a book has appeared in the secondary field. When one beholds in the secondary texts on algebra the heterogeneous arrangement of topics—equations, factoring, exponents, progressions, logarithms, radicals, and imaginaries— one wonders why the attempt was not made long ago to organize secondary mathematics about this principle. Thanks to the earnest recommendation of the National Committee on Mathematical Requirements we shall no doubt soon see numerous attempts to so organize our courses.

In the matter of the construction of a high school course in mathematics let us first give our attention to an outline of the functional relations which might be included in such a course. The first relation to be considered must be of a very simple type. If the work of the seventh and eighth grades has not been organized along the lines which I am proposing I should begin the ninth year with material falling under the type $y = ax$.

Under this heading one can find an abundance of material including such topics as perimeters, measurement and approximate numbers, percentage and interest, scale drawings, similar triangles, trigonometric ratios, uniform motion, motion under constant acceleration, arithmetic progressions, linear variations, and graphs.

After this relation follows quite naturally the general linear type $y = b + ax$. Some of the topics considered under the simpler type may again be taken up. Thus the formula for uniform motion $d = rt$ may be extended to the form $d = d_o + rt$ or $d = d_o + (r_2 - r_1)\,t$. The formula for motion on an inclined plane, a special form of motion under constant acceleration, $v = 32\ t \sin i$, where i is the angle of inclination, may be extended to $v = v_o + 32\ t \sin i$. The formula $h = d \tan a$ which gives the height of an object may be replaced by the formula $h + h_o = d \tan a$. A large number of relations arising from business, social, and physical situations fall under this general linear type.

Under this type one could and probably would give attention to the fundamental operations on directed numbers.

Before passing the linear type we should give some attention to linear pairs. One can find a large amount of material from geometric, social, mechanical, and miscellaneous situations in which two linear relations are concerned.

We come now to the quadratic types. This chapter may be introduced with a consideration of the hyperbolic relation $z = axy$. Under this heading one could consider such topics as areas of parallelograms, triangles, trapezoids, including such formulas as $A = bh \sin a$ and $A = \frac{1}{2}bh \sin a$, work power, the sum of arithmetic series, and the inverse variation. The more general type $z = (x + a)\ (y + b)$ may be considered in connection with the question of the keeping of significant figures in the product of two approximate numbers.

The parabolic relation $y = ax^2$ is rich in applications to geometry and physics. Such topics as the areas of squares and circles, the area under the segment of the line $y = ax$ from o to x, the simple pendulum, work performed by a uniformly variable force such as that performed in stretching a spiral spring,

distance passed over by a body moving with constant acceleration and having an initial o-velocity, projectiles, and the variation as the square. The areas of such figures as the hollow square and the annulus, the area under the line segment $y = ax$ from x_1 to x_2, and the work performed by a uniformly varying force between the values f_1 and f_2 give rise to formulas of the type $y = a\ (x^2 - b^2)$. The type $y = a\ (x + b)^2$ arises from the consideration of changes in x in the relation $y = ax^2$.

The general parabolic type $y = ax^2 + bx + c$ arises in connection with problems on areas, maxima and minima, distances traveled by bodies moving under constant acceleration, and paths of projectiles. The formula for the sum of an Arithmetic Series falls under this type and may be used in solving many interesting problems. This type concludes the work of the ninth grade. The course which I have just outlined is one which I have been developing during the past four years. The problem material which is assumed to lie within the easy comprehension of the pupil arranges itself automatically under the various types of relations.

I have outlined in some detail the work of the ninth grade in order to give you a somewhat definite notion as to the manner in which the notion of functionality may be used by the teacher in ordering the course and in selecting problem material. I shall now speak briefly of the functions which may be considered during the remaining three years.

At the beginning of the second year a little time should be given again to the linear and quadratic functions. Determination of the constants in a linear relation from given data gives a review in the solution of linear equations. The notion of slope of a line and of a curve should be developed. In connection with the slope of a curve one could derive the differential formulas for the quadratic forms and thus study maxima and minima from a new point of view.

The next function in order is the integral rational function

$$y = a_0 + a_1x + a_2x^2 + \cdots a_nx^n.$$

It will first be considered in the simple power forms such as the formulas for volumes of cubes and spheres, or for areas under segments of the simple power curves. In connection with this

general function one might include such topics as polynomial multiplication and division, and the solution of integral rational equations by means of the factor theorem.

Some attention should now be given to the simplest fractional functions and irrational functions. In connection with these functions one would develop the notions of fractional and negative exponents. Now follow the exponential function, logarithms, and the geometric series.

During the first year the trigonometric ratios were introduced. These ratios should now be considered as functions. The various identities should be established and used in solving problems both of the plane and space.

I have not mentioned demonstrative geometry. I presume it should be offered in the high school but it does not fit in the general scheme I have outlined. I certainly should not give more time than one year to both the plane and solid. A large amount of the material presented in the usual course has little or no value since it is not used in the later mathematical work of the student.

Development of the Function Concept in the Mind of the Pupil. Evaluation of the formula. The first part of this paper has been concerned only with the arrangement of material. Let us now see what should be going on in the mathematical development of the pupil. At the beginning of the course he has had practically no experience in mathematical generalization. If he is given the sort of introduction I should give him he would start by translating statements expressed in ordinary language which go over into the simple linear type $y = ax$. The evaluation of y in such relations for various values of x leads him to see that y changes with x and in a very definite manner. Likewise the evaluation of x for various values of y leads to the conclusion that the converse statement holds. Since I have discussed quite fully the use of formulas in ninth grade work in the November number (1921) of the Mathematics Teacher and the development of the function concept in the June number (1921) of School Science and Mathematics I shall limit myself to only a few illustrations in this part of the paper.

Let us consider the formula,

$$h = d \tan \alpha$$

which is used in finding heights. For a fixed α the pupil finds that equal changes in d produce changes in h which are equal though different in value from the changes in d. He can easily discover for himself here that such a relation does not hold for all cases. For if the d is fixed and α varies he finds that equal increases in α do not give similar increases in h. Conversely if h remains fixed a similar statement holds for α and d. Thus, if h is the height of a pole and d is its shadow produced by the sun, his formulas tells him that doubling, or trebling, or quadrupling the angle of elevation does not give corresponding increases in the length of the shadow.

In connection with the consideration of the various types of functional relations the pupil should be given abundant practice in the translation of statements in ordinary language into formulas and in the evaluation of the letters in these formulas. It is this work in evaluation, which, if properly done, will lead him to discover for himself the nature of the various relations.

Thus let him evaluate F in the formula

$$F = 32 + 9/5\ C$$

for $C = -20, -10, 0, +10, +20$,
and evaluate C for $F = -13, -4, +5, +14$.
Let r mean the radius of a circular water-pipe. The area of its cross-section is given by the formula.

$$A = \pi r^2$$

Let him evaluate A for

$$r = 1,\ 2,\ 3,\ 4,\ \ldots$$

Then ask him how the area is affected by doubling the radius, trebling it, and quadrupling it. Had some city fathers, who recently had occasion to double the flow of water from a spring to their city, been given some mathematical training along this line they would not have doubled the diameter of the pipe-line.

As an example of how a pair of linear relations may work together consider the following problem:

Two trains, A and B, are moving east from a city, X. A is 25 miles east of X and is running at a uniform rate of 35 miles

per hour. B is 55 miles east of X and is running at a uniform rate of 30 miles per hour. Find the time when A will overtake B and their distance from X.

The problem may be solved by constructing a table of corresponding values of distances and time as follows:

t	0	1	2	3	4	5	6
d_1	25	60	95	130	165	200	235
d_2	55	85	115	145	175	205	235

As the table is constructed the children see that A is gaining on B and hence will over take B. If the two sets of distances are plotted against the corresponding values of the time and the graphs are constructed the relation between the distances is vividly portrayed.

Geometrical functions. In the course in demonstrative geometry as now generally taught the pupil meets functional relations which he cannot express analytically. For example he can see, although he is not able to state the precise relationship, that a side of a triangle, assuming that the other two sides are fixed, is a fraction of the angle opposite it. As the side grows the angle grows, and conversely. In a later course in trigonometry, or possibly in connection with the chapter on similarity, he may see that this relationship is expressed analytically by the law of cosines. At this time he can answer, and should be required to answer, such questions as the following. What is the length of the side if the angle is 0°? 90°? 180°? Does the side vary as the angle?

In his study of circles he finds that in a fixed circle the length of a chord depends on its distance from the center of the circle. As the distance grows the chord shrinks and conversely. But he cannot say whether the chord varies inversely as the distance, or according to some other law. After he has proved the Pythagorean theorem he can show that

$$C = 2\sqrt{r^2 - d^2}.$$

This formula enables him to answer questions he may have raised as to the nature of the relation.

The chapters on similarity and areas furnish numerous examples of geometrical relationships which can be expressed by simple formulas. These formulas, if studied as indicated above, give the pupil clearer notions as to the nature of these relationships. The character of these relationships may also in some cases be inferred from the geometrical figure. Thus if the sides of a rectangle are doubled it is easy to see that the enlarged rectangle has four times as many square units as the original rectangle.

The graph. A second instrument which is very useful in expressing relationship is the graph. By means of it the general character of the relationship can be seen at a glance. It can therefore be used for predicting the nature of the formula corresponding to it. As an illustration of this point let us plot values of weights used consecutively in stretching a rubber cord and the correcponding lengths of the cord. We find that the points lie approximately on a straight line. We construct a line which seems to fit the points best. After determining the y-intercept of the line and its slope we can write the formula which holds for this particular cord.

On account of its concreteness and its appeal to the eye the graph should be introduced early in the course. It should accompany or possibly precede the formula corresponding to it. The pupil should be able to associate with every type of formula with which he works the corresponding graph and conversely.

Some Reasons Why Functionality Should be Stressed. In the first paragraph of this paper I pointed out the fundamental character of the function concept. Later I showed how the function could be used in effecting the mechanical organization of the course. Finally I sketched a plan whereby the pupil might have his attention directed to the fact that mathematics and science are concerned with functional relations. I wish now to give some reasons why this should be done.

Variables. One of the words of common occurrence in mathematics, and which may be said to be the most distinctly mathematical of all notions,[1] is *variable.* A variable is sometimes

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[1] Russell, Principles of Mathematics, p. 89.

defined as a symbol which in a given discussion may be used to denote any one of a given set of objects. Thus, in the statement, "A horse is a quadruped," the word, horse, is a variable since it is a symbol used to denote any one of a given set of objects. In particular it may be Black Beauty or Man of War.

In the formula,

$$A = \pi r^2,$$

A is a variable since it may refer to the area of any one of a class of objects called circles. r is a variable for a similar reason. Variable is therefore a word of very wide application. In fact the expression of a thought is an expression of relationship between variables.[1] The importance of the variable has been pointed out by Prof. Nunn[2] in the following statement. "The invention of variables was, perhaps, the most important event in human evolution. The command of their use remains the most significant achievement in the history of the individual human being. Ordinary algebra simply carries to a higher stage of usefulness in a special field the device which common language employs over the whole range of discourse. The prudent teacher will, therefore, in the interests of clear understanding and economy of effort, present the technical use of variables in mathematics not as a new thing but as merely a modification of linguistic uses which the pupil mastered, in principle, at his mother's knee."

Professor Judd, in his book on "The Psychology of High School Subjects" in the chapter on language writes: "The world of thought is enormously expanded by the creation and use of words. It is little wonder that man for long ages thought of himself as absolutely distinct from the animal kingdom. Man lives in a world of words; the animals live in a world of things and memories of things. To those who can use words so as to influence the rest of us we give society's great rewards. To the combination of ideas which have been worked out in words we owe changes which have been wrought out in things. In short, our civilization rests on words more than on things themselves, for our civilization differs from primitive uncouth con-

[1] For a fuller discussion of variables see G. A. Bliss on "The Function Concept and the Fundamental Notions of the Calculus" in "Monographs on Topics of Modern Mathematics" edited by J. W. A. Young.

[2] T. P. Nunn, "The Teaching of Algebra," p. 7.

ditions chiefly because the economical methods of thought and action made possible by words have transformed our relation to the world and put at man's disposal forces which could not have been discovered or mastered without the higher modes of abstract thought."

Now the higher modes of abstract thought are concerned with variables and relations between variables. Of course not all the higher modes of abstract thought are concerned with the variables which are employed in elementary mathematics. Here, for the most part, the variable is used to denote any one of a set of numbers. But even so its usefulness extends over a very broad field.

Now the variable of elementary mathematics is usually found in the formula which may always be considered as a functional relation between variables.

The formula. The formula is probably the most important instrument in the expression of mathematical thought. It may also be considered as a compact symbolic expression of a rule or principle. The great advances in mathematics date from the time of the introduction of symbols. Many mathematical formulations expressed in ordinary language would be bewildering and probably useless for practical purposes. Such a formula as Taylor's theorem, viz.,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + ---,$$

is a good illustration. Above I quoted Prof. Judd as saying the "the economical methods of thought and action made possible by words have transformed our relation to the world and put at man's disposal forces which could not have been discovered or mastered without the higher modes of abstract thought." Now may we not say that if common language is an economizer of the first degree the language of mathematics as exemplified in the formula is an economizer of the second degree?

In this connection one writer[1] on philosophy makes the following statement. "The laws of physics, of astronomy, of geology, and the rest are, on the practical side, mere short hand, or rather shortmind, formulae or rules for recovering with a

[1] T. J. McCormack, "Why Do We Study Mathematics?"

minimum of mental labor by means of a little brains and the mechanical manipulation of a pencil, the past and the future facts of nature, which without these rules and laws, we should have indefinite labor, and take indefinite time in recovering. . . . Millions of cases reduce to a single case. All a saving of intellectual labor. But the economy of mental effort is most conspicuous in mathematics. . . . The student who has gained this point of view by contact with mathematical, and especially algebraic study, acquires from it a sense of intellectual power and discipline which can be brought home by no other branch of knowledge."

The importance of the formula can hardly be over emphasized. Formulas are found in great numbers in engineering and technical journals, in the sciences such as physics, chemistry, biology, geology, psychology, social science, pedagogy, in all sciences which have reached the statistical stage, in business, in a vast number of trades and professions in which a mathematical statement or investigation is required. In all of these fields they are used not only for expressing generalizations but for describing the relations between elements in these various fields, that is, for the expression of functionality.

Thus I find listed in a high school physics 64 formulas expressing relationships between various magnitudes in mechanics of solids, liquids, and gases, in sound, heat, electricity, and light. In a geometry I find a list of 101 formulas involving line values, angles, areas, and volumes. In a trigonometry I find listed 56 of the most important formulas pertaining to plane figures and 41 to spherical figures. In books and magazines on engineering I find formulas on almost every page. In a book on stastistical methods applied to education I find a list of 35 formulas. I could go on citing indefinitely from the fields of science illustrations of relationships expressed by means of formulas. In fact one science is considered as having reached a higher stage of development than another if the phenomena with which it is concerned can be explained by a mathematical formulation. Thus physics has reached a higher stage of development than chemistry. Physics made a great advance as a mathematical science when Newton formulated, after having examined the

mass of data collected laboriously by Copernicus, Tycho, Brahe, Kepler, Galileo, their predecessors and co-workers, that the attraction of two material bodies varies as the product of their masses and inversely as the square of the distance between them, i. e.

$$f = \frac{kmm'}{d^2}$$

In the same manner any science advances when a class of phenomena in its field can be explained by a mathematical formula. That is to say when this phenomena can be explained as a relationship between certain variables.

I have shown at some length that the notions of variable and functional relation are vitally related to the advancement of science. I may also add that the advancement of the ability of the individual to think depends on the skill he has attained in seeing and expressing relationships between variables.

Since scientific methods are being employed in more and more fields of human endeavor it is of prime importance that our secondary pupils be given the sort of mathematical training that will fit them for work in these fields. And the sort of mathematical training which will best fit them for the employment of scientific methods is that in which functionality is stressed.

In closing, I quote the following statement from the report of the National Committee on Mathematical Requirements,[1] a statement which should be kept constantly before teachers of secondary mathematics.

"The one great idea which is sufficient in scope to unify the course is that of the *functional relation.* The concept of a variable and of the dependence of one variable upon another is of fundamental importance for everyone. . . . The primary and underlying principle of the course, particularly in connection with algebra and trigonometry, should be the notion of relationship between variables including the methods of determining and expressing such relationship. The notion of relationship is fundamental both in algebra and geometry. The teacher should have it constantly in mind, and the pupils advancement should be consciously directed along the lines which will present first one and then another of the details upon which finally the formation of the general concept of functionality depends."

[1] U. S. Bureau of Education, Secondary School Circular No. 5, p. 3.

DISCUSSION

An historic theorem in plane geometry. I was much interested in Capt. Robert A. Laird's demonstration, page 361 of the October MATHEMATICS TEACHER. It is truly an interesting theorem; but it can be established very much more briefly as follows:

> Dividing equation (1) by equations (2) and (3) and clearing of fractions we obtain the equation,
>
> $$BD : AE : CF = AD : BF : CE.$$

Therefore A, B and C are in a straight line because they are points on the sides (produced) of the triangle DEF, and by the converse of a well known theorem of Menelaus the points are collinear. (If three points are located on the sides of a triangle so that the product of any three segments that do not have a common extremity equals the product of any other three such segments the points are collinear. (See art. 105 Lachlan *Modern Pure Geometry.*)

Similarly it is easily demonstrated that two intersections of the common internal tangents and one intersection of common external tangents are collinear, the six points on three straight lines. Also that the lines through the centers of the circles and the intersections of the internal tangents are concurrent.

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Geometry Speaks. All my life I have suffered from inability to make myself acceptable in the younger set where my brother Arithmetic has been at home for many years. Sister Algebra, in her day, experienced somewhat the same difficulty, but after Arithmetic had taken her to school with him several times in the eighth and ninth grades, people seemed to notice a family resemblance, and while some remained only on speaking terms with her, there were a good many boys and girls between whom and herself a pleasant attachment grew up.

She was really more attractive in personal appearance. Her $\sqrt{\ }$ collars were neat and easy to copy and generally be-

coming, and when it came to writing examples, 6 apples and 10 beans and 5 carrots added to 4 apples and 3 beans and 7 carrots looked more clumsy than

$$\begin{array}{l} 6a + 10b + 5c \\ \underline{4a + \ \ 3b + 7c} \end{array}$$

so that, gradually, she grew to have a place of her own in the list of studies and to become something of a favorite with both teachers and pupils.

Unfortunately I have always seemed to be the oldest and most dignified of our trio. Whether this is due to the matter of association with upper classmen or to a certain formal manner which I have felt obliged to assume in order to keep up the family reputation is uncertain. Whatever the cause, I have always found none too easy the process of introduction to *Senior* High School pupils, even, and have made their acquaintance under heavy odds.

It may be imagined, then, that the prospect of being brought face to face, at different times during the year, with youngsters *just entering their teens* was somewhat appalling. However, all studies are at the mercy of Educators and if Educators say that I must enter the Junior High School into the Junior High School I go—and, having gone, I find what threatened to be an unhappy experience developing more and more into a pleasure.

As a family we three, Arithmetic, Algebra and Geometry, have always lived in wonderful harmony, because, of course, we stand for the same ideals—fidelity to the truth and willingness to see other than our points of view, but, I am bound to confess, we have never worked so effectively as since we all went into business together, so to speak. Everyone knows that we are of the multiform family and have the power of simultaneous existence in widely separated districts. For example, I may, at the same moment, be meeting a class in San Francisco and one in Boston, and we are presented in as many different fashions as there are textbooks and teachers. We enter all classes of intellectual society and Algebra and I are often more pleasantly received than Arithmetic has been. Children sometimes tire of

working with him, year after year, even though those who are fond of him find that their liking increases with length of their acquaintance.

A recent experience with a ninth grade division of children who had not taken very kindly to Arithmetic has interested me and set me to wondering if there may not be some way of making myself sufficiently adjustable and sympathetic to meet pupils a year or even two years younger than these and win them over. There were twenty-seven pupils in this class which, early in September, met for a first lesson in General Mathematics. For two or three weeks I just looked on at recitations while they found out how many dollars (t) apples cost at (f) cents each and what X^2 meant if X meant 5, and how to express the sum of twice A and three times B and what that sum would be if A happened to stand for 7 and B for 3, and how Algebra would tell them the depth of sand in a box, four and a half feet by eight, if a cubic yard of sand was dumped in and leveled off, and matters of that sort.

At first they seemed terribly stupid and I thought the teacher must get tired of saying, "If you had 4 apples and each one cost 5 cents how much would they all cost? Yes. Now, if you had 10 apples, etc. Now, if you had t apples and each one cost f cents how much would they all cost?" She never seemed tired, perhaps because the children acted so differently. Some were cautious, some very venturesome and ready to give any answer, and some so timid that they seemed afraid to speak at all. These timid ones hardly ever had to answer alone until the bold ones had made mistakes and corrected them and the whole class had answered in concert. (I heard the teacher say *that* practice wansn't "according to Hoyle" but children were made before theories, and some men became brave because of the shouts of others) I didn't know what she meant and I didn't care, after a while, because I grew so anxious for the slow ones to get brave enough so that I wouldn't scare them when I came on the scene.

I needn't have worried! They learned my addition, subtraction, multiplication and division axioms in a fortnight, while they were doing some of Algebra's equations, more read-

ily, I thought, than some of the grown-ups learn them. One day, near the end of a period, the teacher began walking around and looking at the children's feet. They were all interested, watching and wondering, until she said, suddenly, "I was looking for a shining pair of shoes. Marion, yours look well. Come out before the class and see if you can place your feet so that they will make an acute angle with each other. Acute means, Class?" "Pointed." "Thin." "Sharp." "Less than a right angle?" were some of the answers. When Marion's feet made an angle Helen was told to put two rulers on the floor, meeting just inside the heels of Marion's boots. Then Alice drew, on the board, two lines making about the same angle which was agreed to be 45 degrees.

There wasn't time for any more, that day, and the teacher didn't say anything about remembering angles, but, a day or two later, she drew a queer looking thing on the board and asked the children if they knew what it was. Some said, "Cape Cod" and she said it was a great comfort to have it recognized because her geography teacher once told her that she would never be able to draw a map so that anyone would know what it represented. Then she drew something to stand for a boat with a man and a boy in it, ran a north and south line through the boat and asked if some one would draw a line meeting that one at an angle of 45 degrees. Half the class wanted to do it, and the lesson stopped after it was done.

Later, in odd bits of time, the children learned to construct a perpendicular, to make an angle equal to a given angle, how to tell and to draw, complementary and supplementary angles and how to select alternate interior and alternate exterior angles. They weren't told many things but allowed to do what they thought right, and if they made mistakes, almost always some one in the class could correct them.

I was particularly interested in a white-faced, mouselike little boy who didn't do anything except stand when the teacher told him to and repeat after her what she said, until it came to constructing equal angles. He got that, right away, and was always the first one called on if company came in. From the time he learned to do this he plucked up his courage and now he

wants to go to the board to try out work in Algebra although, at first, he wasn't on good terms with her, at all.

I'm a member of the General Mathematics Class now, in good and regular standing. The children have learned the four fundamental processes in Algebra, they can multiply binomials by inspection and even factor a little—though they don't know it—they can reduce some fractions to lowest terms, they can make up simple angle problems and solve them and they are almost ready for triangles. A boy in the sloyd class has made a big wooden protractor and, pretty soon, they are going to find the height of the room by "magic" and test the result by measurement.

I have made a long speech and I have made it for this reason. Dressed in my academic costume and with my dignified demeanor on I am a fit associate for people getting ready for college, but, dressed in my play clothes and allowed to have a good time with Junior High School youngsters I think I can both give them pleasure and fit them, without their suspecting it, for real work, later on. Why not try letting me play with them and see if I am not right? EVA M. PALMER,

High School, Winchester, Mass.

Notes on the Teaching of Arithmetic. For years we have given the child special help and devices in reading. He has had his families of words, his phonics, his charts, and illustrated books. In arithmetic what have we done? Tried the topical plan, the spiral plan, and other plans. Given some games and drills and helps to make the work more attractive, but never has the real cause for failure been reached. We have done nothing to remove the drudgery attendant on acquiring the mechanics of the subject. The natural combination of numbers, the aids to computation, the beauties of the number system have been kept away from the pupil and from many teachers. Why not make a study of arithmetic?

To be successful in arithmetic get away from drudgery, avoid too cumbersome computing, study the number system, make arithmetic a joy. You can't get speed and accuracy with long computations. And you can't get problem work until the pupil has facility in number work.

Our number system is so beautiful, so easy, so full of possibilities that if only a small amount of time is given to it your number work will improve.

You must consider—the order in which number is taught,
—the composition, decimal and factor.

Begin with doubles: $1 + 1$, $2 + 2$, etc.

Then use doubles plus 1, plus 2, etc.

An attractive chart has been developed for this work.

$2 + 2 = 4$, then $2 + 2 + 1 = 5$, or $2 + 3 = 5$.

Use the decades: $2 + 2$, $12 + 2$, $22 + 2$,
also $12 = 10 + 2$.

Emphasize this continually. It helps all along the line.

$15 \times 12 = 15(10 + 2)$. Introduce some areas. (Oral).

Add beginning at the left: 25, 16, 41, 17, 20, 30, 70, 80, 85, 91, 92, 99.

Change the order of teaching tables. There is no reason why they should be taught in consecutive order. After some preliminary work, teach 5,s, 10's, and 20's. Your pupil knows many of the facts. The grocery store and candy counter have been his teachers. Incidentally the boy feels pretty big when he can handle 20's, and unconsciously he learns his 2's.

Then teach 2's. 4's and 8's naturally follow. Reverse each pair of factors. Teach 1/4, 1/2, 1/8. Use Dry Measure for your application. This table is close to the pupil's experience. Reversing factors teaches other tables.

Next teach 3's, 6's, 12's. Teach 1/3, 1/6.

9 is a hard table for many children. It is all done now except 7×9 and 9×9. Teach 9's.

Use the yard for concrete work in 3's, 6's, and 9's, 12's.

Teach 7's. This is all done except 7×7 and 7×11. Use the week for application.

Teach 11's. This is all complete except 11×11.

Don't be afraid to drill. Lessen your work as teacher and increase the efficiency of your pupil by giving keyed examples, no two alike. This multiplies the drill work in fundamental processes and prevents copying. Key all examples which you give

for speed and accuracy. Such speed work applies to Addition, Subtraction, Multiplication, Division, Mixed Numbers, Decimals, at board or seat.

Solve:

$1680175 \div 321$

$$\begin{array}{r} 864 \\ 941 \\ \underline{582} \end{array}$$

These are keyed by two digits. You can use other keyed schemes also. The teacher can make such examples or purchase them.

To check division, add the subtrahends that have been checked.

Teach squares and cubes.

Teach the relations of odd and even numbers in series. There are many interesting laws connected therewith.

Make use of factors constantly. Express a solution, delaying computation as long as possible.

Make use of architects' triangles. Take advantage of $\sqrt{2}$ and $\sqrt{3}$.

Use factors in reduction of fractions, even in decimals. Make use of the fact that the denominater of a decimal fraction is a power of 10.

Apply decimals to interest. Use no other method.

By the above means, so briefly outlined, our pupils will learn not only to compute but to think. Their high school mathematics will no longer suffer.

R. L. SHORT.
St. Louis, Mo.

NEWS AND NOTES

President John H. Minnick and the Executive Committee of the National Council of Teachers of Mathematics have perfected plans to present the interests of the Council to practically all organizations of mathematics teachers in the United States. State Representatives have been appointed, as follows:

Alabama—Frank Ordway, Central High School, Birmingham
Arizona—A. L. Hartman, Mesa Union High School, Mesa
Arkansas—George W. Drake, University of Arkansas, Fayetteville
California—Gertrude E. Allen, University High School, Oakland
Colorado—E. L. Brown, Northside High School, Denver
Delaware—Mrs. Elinor B. Rosa, Milford
District of Columbia—Harry English, Board of Examiners, Washington
Florida—Miss Olga Larson, Box 84, Apopka
Georgia—George W. Brindle, Surrency
Idaho—Winona M. Perry, 719 Sherman Ave., Couer D'Alene
Illinois—R. L. Modesitt, 1703 S. 7th St., Charleston
Indiana—Walter G. Gingery, Shortridge High School, Indianapolis
Iowa—Ira S. Condit, Iowa State Teachers College, Cedar Falls
Kansas—Miss Inez Morris, 728 State St., Emporia
Kentucky—V. D. Roberts, Somerset
Louisiana—Jeanne Vautrain, 1820 N. Rampan St., New Orleans
Maine—E. L. Moulton, Edward Little High School, Auburn
Maryland—Miss N. V. Orcutt, Girls' Latin High School, Baltimore
Massachusetts—William H. Brown, High School, Amherst
Michigan—John P. Everett, Western State Normal School, Kalamazoo
Minnesota—W. D. Reeve, 828 University Ave., Minneapolis
Mississippi—Miss Clyde Lindsey, Oxford
Missouri—Charles Ammerman, McKinley High School, St. Louis
Nevada—Miss Bertha C. Knemyer, Elko Co., High School, Elko
New Mexico—T. C. Rogers, 1018 Fourth St., E. Las Vegas
New York—Raleigh Schorling, 423 West 123rd St., New York City
Ohio—Miss Florence M. Brooks, Fairmount, Jr. High School, Cleveland
Oklahoma—C. E. Herring, Box 489, Oklahoma City
Oregon—Florence P. Young, Frankline H. S., Portland
Rhode Island—P. S. Crosby, 110 N. Bend St., Pawtucket
South Carolina—J. Bruce Coleman, University of South Carolina, Columbia
South Dakota—Iona J. Rehn, 735 S. Summit Ave., Sioux Falls
Tennessee—F. L. Wrenn, McCallie School, Chattanooga
Texas—J. O. Mahoney, 1900 Crockett St., Dallas
Vermont—Llewellyn R. Perkins, 6 Franklin St., Middlebury
West Virginia—Miss Blanche Stonestreet, 591 Spruce St., Morgantown
Wisconsin—Miss Mary A. Potter, Racine High School, Racine

These representatives are actively engaged in urging the teachers in their respective states to affiliate with the Council, and to participate, in a more direct way, in the reorganization movement now being effected in mathematical education. A special circular has been prepared to set forth the purposes and

values of the Council. Copies may be secured from your representative, from Mr. John A. Foberg, Secretary-Treasurer, Camp Hill, Pa., or from President John H. Minnick, School of Education, University of Pennsylvania, Philadelphia.

Watch the January MATHEMATICS TEACHER for the program of the fourth annual meeting of the National Council of Teachers of Mathematics, to be held in Cleveland during the week of the Department of Superintendence.

The first meeting of the Chicago Mathematics Club for the season 1922-1923 was held on the evening of October 13, at the Central Y. M. C. A., 19 S. LaSalle Street. After dinner Pres. H. C. Wright called the meeting to order and appointed a nominating committee. The committee proposed the following officers for the year 1922-1923 who were unanimously elected:

Everett W. Owen, Oak Park, President.

Olice Winter, Harrison Tech., Secretary-Treasurer.

Edwin W. Schreiber, Proviso, Recording Secretary.

The tentative program for the next three meetings is as follows:

November meeting—The Slide Rule.

December meeting—Field Problems.

January Meeting—Results of Classification in Mathematics Classes.

Several other business matters were brought before the Club and disposed of, before our new President, E. W. Owen, gave a very fine summary of the articles by Prof. Edward L. Thorndike which have been published in the last five issues of the MATHEMATICS TEACHER. An interesting discussion followed Pres. Owen's remarks.

The Mathematics Section of the New York State Teachers' Association was held at Syracuse, November 28. Professor Wilfred H. Sherk, of Buffalo, arranged the following program:

The Contacts of Mathematics with the World of Affairs, Ernest H. Koch, Jr., Group Adviser, Technical High School, Brooklyn, N. Y.; *The Mathematical Puzzle as a Pleasant Stimulus to Serious Work,* Professor Walter B. Carver, Cornell Uni-

versity; *The Psychology of Transfer in Mathematics,* John R. Clark, The Lincoln School of Teachers College; *The Training of Teachers,* Professor J. W. Young, Dartmouth College; *Discussion of the Report of the Committee of 21,* Charles F. Wheelock, Assistant Commissioner of Education, New York State Department of Education; *A Study of the Problems in Arithmetic,* William Alexander Smith, Hackensack, N. J.; and *A Survey of Problems in the Teaching of Mathematics,* William Betz, Supervisor of Mathematics, Rochester, N. Y.

REVIEWS

The Thurstone Vocational Guidance Tests: Arithmetic, Algebra, Geometry—The Thurstone Vocational Guidance Tests constitute a series of five tests, one for each of the following subjects: Arithmetic, Algebra, Geometry, Physics and Technical nformation. These five tests are designed to be used together to test high school seniors and college freshmen to determine their probable success in an engineering college.

A committee appointed in accordance with a resolution adopted by the society at its Baltimore meeting in 1918, proposed a co-operative research of the diagnostic value of various forms of entrance tests. For this purpose Dr. L. L. Thurstone, chairman of the committee, compiled six tests—the present series of five vocational guidance tests and an intelligence test. These six tests were given to about 8,000 freshmen engineering students in forty-three engineering colleges shortly after their admission. Records of the tests were filed and later compared with the students' scholarship performance. A comparison made by the committee between these test records and scholarship performance shows that the tests have a distinct predictive value.

The tests are the outgrowth of the efforts of the society for the Promotion of Engineering Education to obtain a comparison of students' scholarship in engineering courses with their ratings obtained on psychological, objective, trade, or similar tests given at their admission to college.

The items in each of the tests are all selected as having a direct appeal to students with engineering interests. The items are so constructed that their solution calls for a minimum of computation and a maximum of reasoning, which is the important factor in success in engineering and gives the best indication of engineering interest as distinct from interest in figures only.

The Arithmetic Test consists entirely of problems. The Algebra Test contains, in addition to problems, exercises designed to test skill in algebraic technique. The Geometry Test requires the making of actual geometrical constructions with compass and straight edge. No formal proofs are required.

[1] Published by the World Book Company, Yonkers-on-Hudson, New York.

The items are arranged in approximate order of difficulty. Space is provided for answers to appear in columns. Every answer is either right or wrong; no partial credits are given. With the exception of the Geometry Test, which must be scored by a geometrician, the scoring is entirely objective, requiring no judgment, and can therefore be done by clerks. All directions are given before the examinee begins. The time limit of each test is thirty minutes, and the work is uninterrupted.

A number of tables and charts are given in the Manual of Directions, in order that a student's score may be compared with the scores made by the 8,000 students who have taken this test and his relative standing among these students in engineering ability, as predicted by the tests, be determined.

A careful statistical study was made of the degree to which success in engineering courses would be predicted by means of each of the various tests and by records of high school achievement in physics, chemistry, geometry, algebra and English. Coefficients of correlation were obtained between scholarship in the engineering college and scores in the tests and between scholarship in the college and high school achievement. These correlations show that on the average the tests of this series have an appreciably higher predictive value than the records of school achievement. The relative value of the test scores and school achievement in predicting engineering scholarship is shown in a table in the Manual. A table is also given showing the probability of the success of a student coming within the highest quarter, second quarter, third quarter, or lowest quarter of scores in each test.

On the whole, the tests have demonstrated their distinct value as an adjunct to other available data, such as high school scholarship, and may be employed to determine with a high degree of accuracy the probable success of a student in college engineering courses and in engineering professions.

ARTHUR S. OTIS,
Yonkers-on-Hudson